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FILTERING AND SMOOTHING OF TIME RELATED DATA

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6 FILTERING AND SMOOTHING OF TIME RELATED DATA.

by

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ABSTRACT

This paper discusses several approaches to handling the filtering and smoothing of time related data. The specific case addressed is the Antitank Missile Test (ATMT) conducted at US Army Combat Developments Experimentation Command using the Range Measuring System. Methods discussed include difference quotients, Kalman filters, spline functions, least squares approximation, convolution filters, and minimum maximum error techniques.

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FILTERING AND SMOOTHING OF TIME RELATED DATA

by David W. Bash, PhD

1. INTRODUCTION.

a. This paper discusses several approaches to handling data recorded by the Range Measuring System (RMS) at US Army Combat Developments Experimentation Command (CDEC) to obtain velocity and acceleration estimates for vehicles. The discussion also applies to any data when approximate X, Y, Z coordinates as a function of time are given.

b. As in most data handling situations, the computation of velocity and acceleration is highly dependent upon the physical setting, the accuracy of the measurements, and the desired application of the results. This paper was prepared as a result of investigations made during analysis of the Antitank Missile Test (ATMT) conducted at CDEC and is illustrated with data from that study. The ATMT analysis was interested in following the motions of the vehicle closely in order to ascertain the effects of evasive maneuvers of targets on certain weapon systems.

c. The accurate position, velocity, and acceleration of a vehicle may be required for many test purposes. This paper shows that some approaches to calculating velocity and acceleration yield erroneous results. ATMT used RMS data in terms of time, X-coordinate, Y-coordinate, and Z-coordinate for a variety of target vehicles traversing several general paths on different terrains and following various operating instructions. RMS data give the X-, Y-, and Z-coordinates in meters approximately 10 times per second. Usually, but not always, the time differences between consecutive measurements in ATMT were 0.1 ± 0.01 seconds. The error in coordinates was reported to have a standard deviation of 1 meter, a mean of zero, and a normal distribution. However, the data, as received, had already been processed by a Kalman filter so that actually a maximum likelihood estimate of the true location was being reported. Thus, the three major errors in computing the velocity and acceleration of the vehicle were the following:

(1) The data error in location, referred to in this paper as the inherent error.

(2) The error in computer computation, referred to as the roundoff error.

(3) The error by using a finite approximation formula, referred to as the truncation error.

d. Much of the discussion in the following paragraphs is with respect to just one coordinate. However, all coordinates need to be considered, and the magnitude and direction of velocity and acceleration vectors found, since maneuvers may impart more velocity and acceleration than the vehicle can perform in a straight line. This paper discusses the strengths and weaknesses of several methods for application. The references listed at the end of the paper give more complete implementation information; the algorithms included here are intended merely to illustrate the various methods.

2. DIFFERENCE QUOTIENT APPROACH.

a. Standard Approach. The first technique discussed is the straightforward approach of using the definition of first and second derivatives to compute difference quotients. Table 1 presents a section of the results of approximating the first derivative of $x(t_i)$ by:

$$x'(t_i) = \frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i} \qquad \text{(Eq 1)}$$

and the second derivative by:

$$x''(t_i) = \frac{x'(t_{i+1}) - x'(t_i)}{\dfrac{t_{i+2} - t_i}{2}} \quad . \qquad \text{(Eq 2)}$$

Table 1. Standard difference quotients

t_i: (sec)	$x(t_i)$: (meters)	$x'(t_i)$: (meters/sec)	$x''(t_i)$: (meters/sec^2)
37010.76	55195.9	5.00	25.25
37010.82	55196.2	7.78	-44.44
37010.91	55196.9	4.44	17.28
37011.00	55197.3	6.00	- 5.74
37011.10	55197.9	5.45	0.96

Accelerations larger than straight-line speedup or braking could occur in a sharp turn, but values of at least 4 g (one g is the acceleration due to gravity of about 9.8 meters per second^2) are unreasonable for the vehicles being considered, as are changes in acceleration of more than 7 g in one-tenth of a second. The main source of error in this approach is the inherent error. A small error; e.g., one-half standard deviation or one-half meter in just one $x(t)$ value of the first two entries in table 1, could cause 100 percent error in the $x'(t)$ value. Actually, the inherent error in the acceleration is proportional to the error in the data divided by the time difference squared, roughly a factor of 100 times the data error.

b. Point Skipping Approach. One approach to controlling the error difference quotients would be to increase the time difference between consecutive points (i.e., skip some data points) to hold down the inherent error. A method like this could be sound, at least heuristically. High frequency random measuring errors may be prevented from affecting the differences used in computing difference quotients. However, in an application interested in evasive maneuvers, no predetermined, large-size skipping rule can guarantee that evasive maneuvers are not missed. Table 2 displays the difference quotients for a small-size skipping rule. Again, these values are too rough to use in most applications. Another source of error is that, since the velocities and accelerations are not constant functions of time, the truncation error is proportional to a power of the time difference (see chapter 4 of Ralston, reference 8). Thus, increasing the time differences may increase the truncation error.

Table 2. Skipping points difference quotients

t_i (sec)	$x(t_i)$ (meters)	$x'(t_i)$ (meters/sec)	$x''(t_i)$ (meters/sec^2)
37010.76	55195.9	5.78	- 1.93
37011.21	55198.5	4.81	- 4.68
37011.75	55201.1	2.50	0.00
37012.23	55202.2	2.50	- 3.50
37012.79	55203.7	0.68	0.82

c. Interpolation. Another approach is to interpolate the data with some type of function and compute the first and second derivatives of the function. Usually, a polynomial is tried (e.g., Lagrange polynomials, Hermite polynomials), but rational functions or transcendental functions possibly could be used. One trouble with this approach is in choosing the degree of other parameters in the expression. If the degree is low (i.e., only a few points at a time are interpolated), the polynomials will follow the errors of the data and the derivatives of the polynomials will be similar to the difference quotients discussed previously. If the degree is high (i.e., many points at a time are interpolated) the polynomials will more than likely have extreme and unreasonable variations (see page 354 of Hamming, reference 6). Again, one might try employing only selected data points, but the same problems occur here as in the point skipping approach. Thus, these direct approaches almost always yield questionable results.

3. KALMAN FILTER. Subparagraphs 3a and 3b, below, discuss the use of the Kalman filter in filtering; i.e., changing the data point by point as they are presently being received or considered. Subparagraph 3c discusses smoothing; i.e., looking at and changing past values.

a. Filtering. If the errors are distributed normally with mean zero, the Kalman filter, which is a weighted-least-squares estimate of the true coordinates, gives a maximum-likelihood or minimum-variance estimate of the true location. Matrix vector notation, with a superscript T for the transpose, is used in the following discussion. Let:

$\underline{x}$ = actual position vector

$\underline{z}$ = measured position vector

$\bar{\underline{x}}$ = estimated position vector

H = a 3x3 matrix stating the assumed relation of actual to measured

$\underline{v}$ = random error vector

$$\underline{z} = H\underline{x} + \underline{v} \qquad \text{(Eq 3)}$$

$R(\underline{v})$ = a 3x3 variance-covariance matrix for $\underline{v} = E((\underline{z} - H\underline{x})(\underline{z} - H\underline{x})^T)$

M = a 3x3 variance-covariance matrix for $\underline{x} - \bar{\underline{x}} = E((\underline{x} - \bar{\underline{x}})(\underline{x} - \bar{\underline{x}})^T)$

The objective in Kalman filtering is to find $\hat{\underline{x}}$ as the value of $\underline{x}$ that minimizes the quadratic form:

$$J = \tfrac{1}{2} \left[(\underline{x} - \bar{\underline{x}})^T M^{-1} (\underline{x} - \bar{\underline{x}}) + (\underline{z} - H\underline{x})^T R^{-1} (\underline{z} - H\underline{x}) \right] \qquad \text{(Eq 4)}$$

The $\hat{\underline{x}}$ is found by standard differential arguments. Note that errors $\underline{x} - \bar{\underline{x}}$ and $\underline{z} - H\underline{x}$ are weighted by the prior expected values of $\overline{(\underline{x} - \bar{\underline{x}})(\underline{x} - \bar{\underline{x}})^T}$ and $\overline{(\underline{z} - H\underline{x})(\underline{z} - H\underline{x})^T}$, respectively.

b. Multistage Filtering. The situation is more complicated in multistage processes, as exemplified by the ATMT study. In the following discussion, the notation is changed slightly to accommodate a linear first order prediction step for one model of RMS data. Assuming the best least-squares estimate for the (i - 1)th point, the best least-squares estimate for the ith point is stated. Let:

$$\hat{\underline{x}}_i = \text{best least-squares estimate for position-velocity vector}$$

$$= (x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i)$$

$$\Phi_{i-1} = \text{transition matrix from state of time } t_{i-1} \text{ to time } t_i$$

$$= \begin{pmatrix} I_3 & I_3 dt_i \\ 0 & I_3 \end{pmatrix}, \quad I_3 \text{ a 3x3 identity matrix}$$

$$H_i = \begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{a 6x6 matrix}$$

$\underline{v}_i$ = random error vector (Eq 5)

$\underline{z}_i$ = measured position vector = $(x_i, y_i, z_i, 0, 0, 0) = H_i\underline{x}_i + \underline{v}_i$

R_i = expected value matrix for $(\underline{z} - H_i\underline{x}_i)(\underline{z}_i - H_i\underline{x}_i)^T = \underline{v}_i\underline{v}_i^T$

$\bar{\underline{x}}_i = \Phi_{i-1}\hat{x}_{i-1}$ = predicted position-velocity vector

$$M_i = \Phi_{i-1} P_{i-1} \Phi_{i-1}^T \qquad \text{(Eq 6)}$$

$$P_i = (M_i^{-1} + H_i^T R_i^{-1} H_i)^{-1}$$

M_0 = expected value matrix for $(\underline{x} - \bar{\underline{x}}_0)(\underline{x} - \bar{\underline{x}}_0)^T$

$$K_i = P_i H_i^T R_i^{-1} \qquad \text{(Eq 7)}$$

Then the best least-squares estimate is a standard differential change predicted value $\bar{\underline{x}}_i$ plus a correction term proportional to the difference between measured and predicted value:

$$\hat{\underline{x}}_i = \bar{\underline{x}}_i + K_i(\underline{z}_i - H_i\bar{\underline{x}}_i),\ i \geq 1. \qquad \text{(Eq 8)}$$

For computational purposes, it is helpful if the dt_i's are small compared to 1. Then the assumption is made that $dt_i = 0$, which implies that Φ_{i-1} is the identity in equation 6. Also, it is assumed that the errors in the coordinates are independently and identically distributed so that M_i and P_i are all diagonal. Then $\hat{\underline{x}}_i$, with these assumptions, is only approximately the maximum likelihood estimate. However, these assumptions may be needed to speed up calculations to run in real time, as was desired in the original filtering of the ATMT data (reference 3). Chapter 12 of Bryson and Ho (reference 4) contains the details of the above derivation, and the BDM documentation (reference 3) includes the practical simplifications. Reference 2 describes a similar model accommodating acceleration, and chapter 4 of the text by McGarty (reference 7) gives another readable account. Table 3 is a partial output from a programmed version of the algorithm in the McGarty text.

Table 3. Kalman filter example

t_i: (sec)	$x(t_i)$: (meters)	$x'(t_i)$: (meters/sec)	$x''(t_i)$: (meters/sec^2)
34130.13	56464.9	6.06	1.47
34130.24	56465.5	6.22	1.48
34130.33	56466.2	6.35	1.47
34130.43	56466.7	6.45	1.45
34130.59	56467.4	6.61	1.46

c. Smoothing. After all the data have been recorded and filtered, then one may start with the last value $\underline{x}_N$ and work forward, smoothing the data by minimizing a similar but more complicated functional. The results that follow, and some necessary initializing conditions, are derived in chapter 13 of Bryson and Ho. Let:

$\hat{\underline{x}}_{i/N}$ = the best least-squares estimate to i^{th} point given N points. (Eq 9)

Then:

$$\hat{\underline{x}}_{i/N} = \hat{\underline{x}}_i - C_i\,(\bar{\underline{x}}_{i+1} - \hat{\underline{x}}_{i+1/N}) \qquad \text{(Eq 10)}$$

where:

$$C_i = P_i \Phi_i^T M_{i+1}^{-1}. \qquad \text{(Eq 11)}$$

The objective in Kalman filtering and smoothing is to change the dependent variable values to be nice in a statistical sense. Knowledge of the variance-covariance matrices discussed previously is necessary before doing Kalman filtering and/or smoothing. Often, the variances are adjusted to yield nice output.

4. POLYNOMIAL SPLINE FUNCTIONS.

a. Standard Approach. Spline function approximation is the interpolation of data by polynomials of a given degree, each polynomial defined on distinct intervals between consecutive data points. These polynomials are determined so that the resulting function defined on the entire domain has continuous derivatives of all orders up to but excluding the degree of the polynomial (even at the data points). A property of interest in spline approximation is that, using polynomials of degree 2k-1, the integral:

$$\int_a^b (f^{(k)}(t))^2 dt \qquad \text{(Eq 12)}$$

is a minimum over all functions with the same interpolation and smoothness property. In the case of cubic polynomials (the most often used degree) the second derivative, or accelerations, is minimized in this sense. This property of minimal acceleration may not be too severe even if the project is mainly interested in actual accelerations. The reasons include the fact that the function must have sufficiently large second derivative in order to interpolate the data points and most physical systems operate in a conservation of work (proportional to acceleration) principle. The spline approach can be advantageous if the values of the first and second derivatives are needed at points other than data points. In the derivation of these polynomials nice computational formulas are developed for the derivatives at any domain value (subparagraph 4b, below). This capability might be needed if the data points are rather sparse compared to RMS data and the unreasonable variations mentioned in paragraph 2 are to be avoided. However, in order to obtain some of the parameters in the formulas, it is necessary to invert an n by n matrix where n is the number of data points being fitted. The matrix is fairly diagonal dominant, but care must still be used to keep the roundoff error under control if the number of points is large. These problems can be alleviated if the fitting is done in segments or if only selected points are used; however, the first approach may destroy some desired continuity and the second may miss some evasive maneuvers as discussed in subparagraph 2c. Also, the inherent error may become serious. The algorithms deriving the parameters essentially compute first and second order difference quotients and thus are open to the criticisms raised in subparagraphs 2a and 2b.

b. Formulas for Cubic Spline Interpolation. Given the following:

$$t_i, \ i=1, \ldots, n$$

$$x_i, \ i=1, \ldots, n$$

$$h_i = t_{i+1} - t_i, \ i=1, \ldots, n-1 \qquad \text{(Eq 13)}$$

the following assumptions are made:

$$x(t) = x_i(t),\ t_i \le t \le t_{i-1},\ i=1, \ldots, n-1$$

where the $x_i(t)$ are third degree polynomials in t

$$x_1(t_1) = x_1 \qquad \text{(Eq 14)}$$

$$x_i(t_i) = x_i = x_{i-1}(t_i) \quad i=2, \ldots, n-1$$

$$x_{n-1}(t_n) = x_n$$

$$\dot{x}_{i-1}(t_i) = \dot{x}_i(t_i) \quad i=2, \ldots, n-1$$

$$\ddot{x}_{i-1}(t_i) = \ddot{x}_i(t_i) \quad i=2, \ldots, n-1$$

The functional values, i=1, . . . , n - 1, are:

$$\ddot{x}_i(t) = c_i\left(\frac{t_{i+1} - t}{h_i}\right) + c_{i+1}\left(\frac{t - t_i}{h_i}\right) \quad t_i \le t < t_{i+1}$$

$$\dot{x}_i(t) = \frac{x_{i+1} - x_i}{h_i} - \frac{h_i}{6}\left[c_i \left\{-1 + 3\left(\frac{t_{i+1} - t}{h_i}\right)^2\right\} + c_{i+1}\left\{1 - 3\left(\frac{t - t_i}{h_i}\right)^2\right\}\right] \quad t_i \le t < t_{i+1}$$

(Eq 15)

$$x_i(t) = x_i\left(\frac{t_{i+1} - t}{h_i}\right) + x_{i+1}\left(\frac{t - t_i}{h_i}\right)$$

$$- \frac{h_i^2}{6}\left[c_i\left\{\left(\frac{t_{i+1} - t}{h_i}\right) - \left(\frac{t_{i+1} - t}{h_i}\right)^3\right\} + c_{i+1}\left\{\left(\frac{t - t_i}{h_i}\right) - \left(\frac{t - t_i}{h_i}\right)^3\right\}\right]$$

$$t_i \leq t < t_{i+1}$$

$$\ddot{x}_{n-1}(t_n) = c_n$$

$$\dot{x}_{n-1}(t_n) = \frac{x_n - x_{n-1}}{h_{n-1}} + \frac{h_{n-1}}{6}(c_{n-1} + 2c_n)$$

$$x_{n-1}(t_n) = x_n$$

where the c_i, $i=1, \ldots, n$ are the solutions to:

$$AC = D \qquad \text{(Eq 16)}$$

The entries in the matrix A and vector D are:

$$A = (a_{ij})$$

$$i, j = 1, \ldots, n$$

$$D = (d_i)$$

$$a_{11} = 2h_1 \qquad a_{i,\,i-1} = h_{i-1}$$

$$a_{12} = h_1 \qquad a_{i,i} = 2(h_{i-1} + h_i)$$

$$i=2, \ldots, n-1$$

$$a_{n,n-1} = h_{n-1} \qquad a_{i,i+1} = h_i$$

$$a_{n,n} = 2h_{n-1} \quad \text{rest } a_{ij} \text{ are zero}$$

$$d_1 = 6\left(-J + \frac{x_2 - x_1}{h_1}\right)$$

$$d_n = 6\left(K - \left(\frac{x_n - x_{n-1}}{h_{n-1}}\right)\right)$$

$$d_i = 6\left(\left(\frac{x_{i+1} - x_i}{h_i}\right) - \left(\frac{x_i - x_{i-1}}{h_{i-1}}\right)\right) \qquad i=2, \ldots, n-1$$

J and K are approximations to $\dot{x}(t)$ at t_1 and t_n, respectively. Further information may be obtained in chapter 20 of Hamming. Figures 1, 2, and 3 show the plots of position, velocity, and acceleration, respectively, using the cubic spline approach to smoothing using every eleventh data point. Note the acceleration is piecewise linear and continuous as it should be but still is realistic.

c. Spline Approximation with Least Squares Approach. This approach is designed to minimize the inherent error problem mentioned in the previous paragraph. The usual approach minimizes the weighted sum of a smoothness term and the sum of the squares of the deviations between observed values and smoothed values. Let:

$$E(x) = \sum_{i=1}^{n} (x(t_i) - x_i)^2 \qquad \text{(Eq 17)}$$

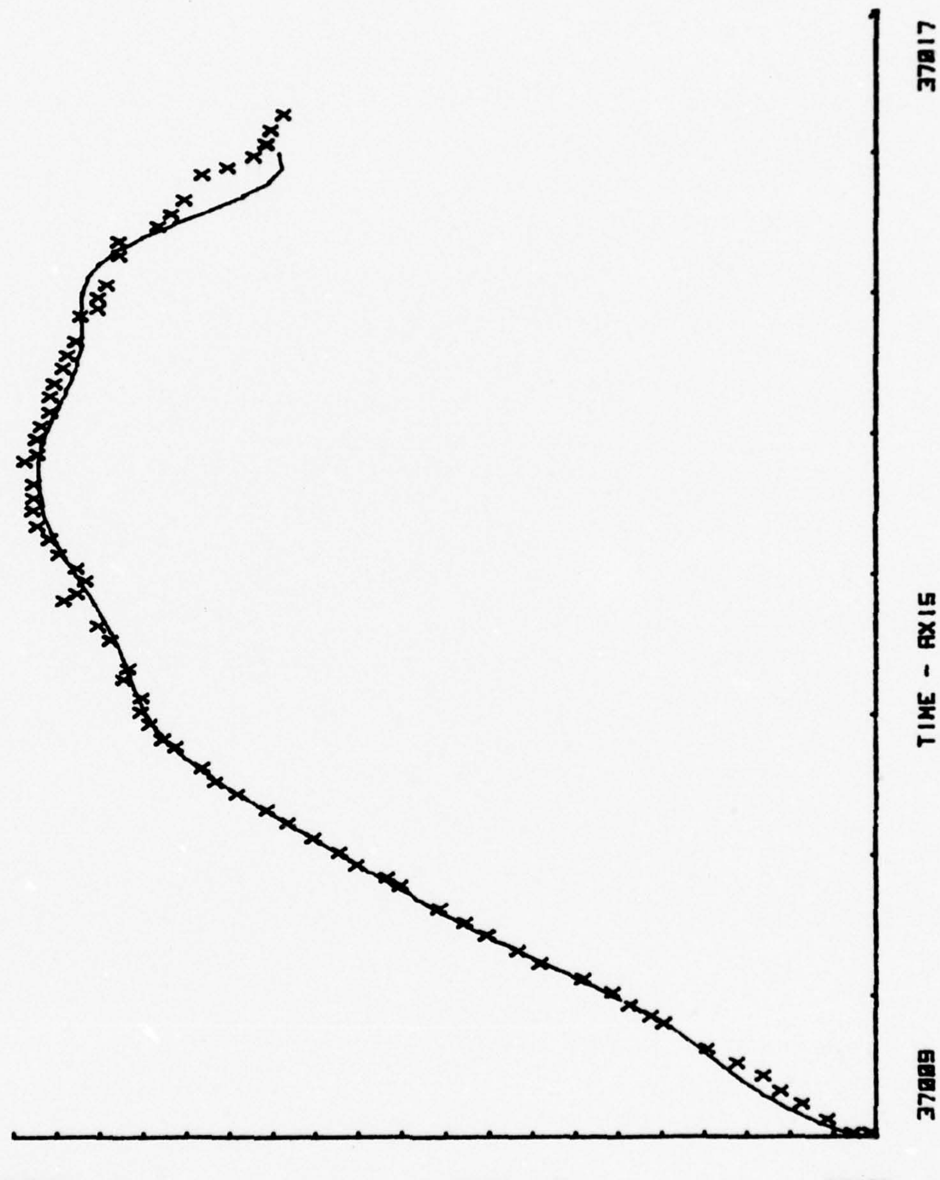

Figure 1. Position plot

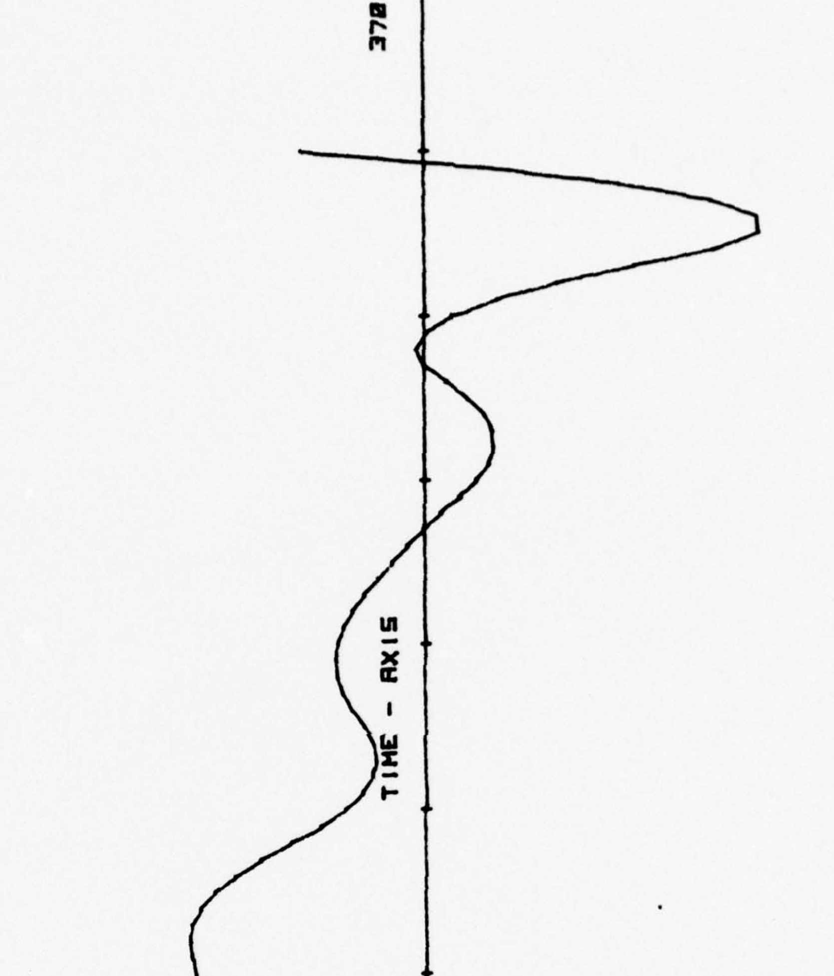

Figure 2. Velocity plot

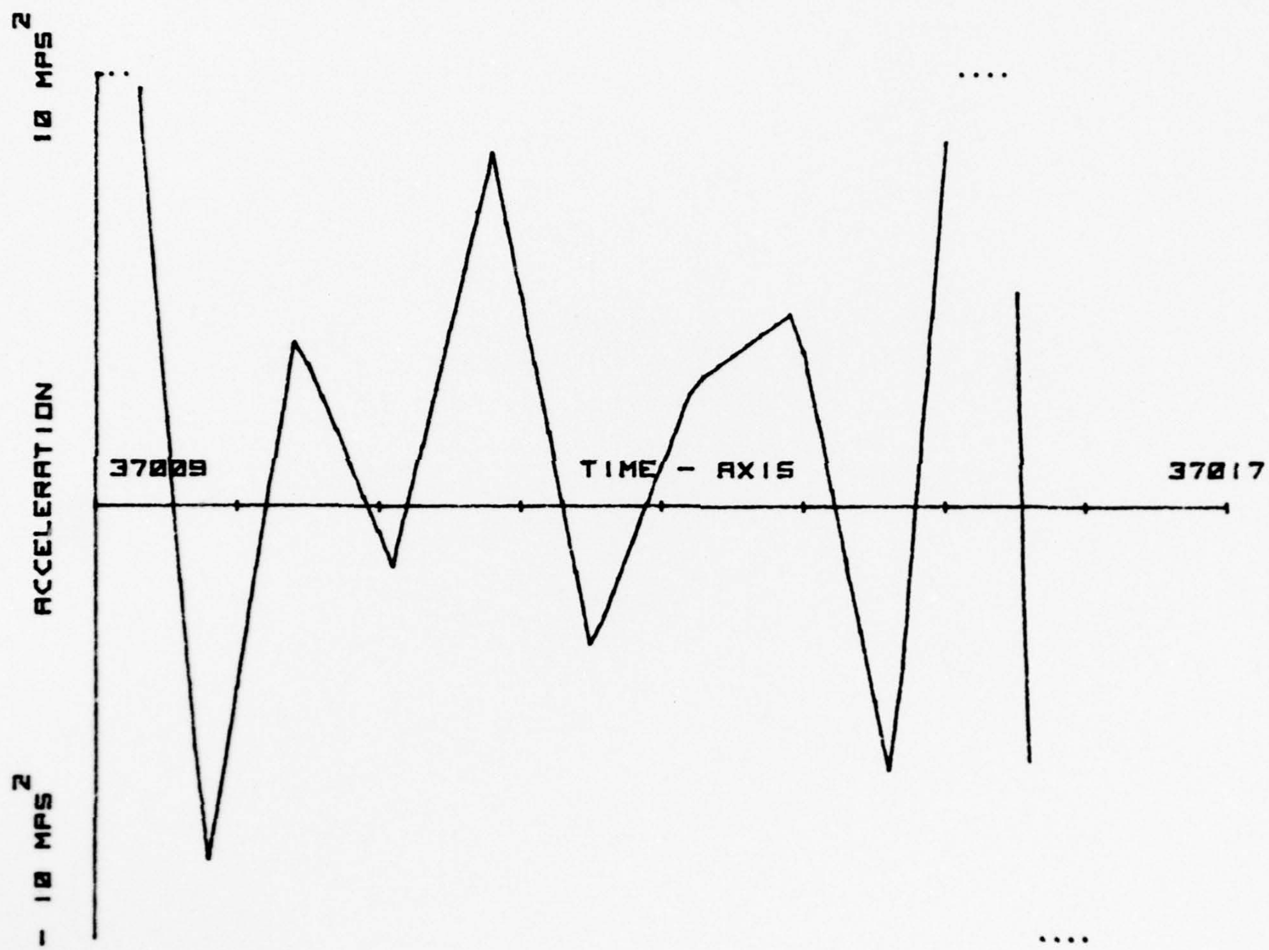

Figure 3. Acceleration plot

$$S(x) = \int_a^b (x^k(t))^2 \, dt \qquad \text{(Eq 18)}$$

$$n = 2k - 1, \; k \text{ integer} \qquad \text{(Eq 19)}$$

$$0 \leq \rho \leq 1$$

Then the smoothing spline function $x(t)$ for the data points (t_i, x_i), using polynomials of degree n, is the function that minimizes:

$$S(x) + \rho E(x) \qquad \text{(Eq 20)}$$

The factor ρ weights the relative importance of the two terms. Given ρ, to solve for $x(t)$ it is required to solve a system of equations similar to the systems in paragraphs 4a and 5c. Hence, the cautions with respect to computation apply here. However, there is no interpolation now, but the first and second derivations will be smoother. Some of the reasons that this method may not be desirable as applied to RMS data are the fact that derivatives at non-data points are not needed and there are too many points to use all at one time, requiring doing the analysis in segments (destroying some continuity) or using selected points (raising concern about missing a maneuver). More detail on implementation may be found in Schumaker, reference 10.

5. LEAST SQUARES APPROXIMATION (FILTERS).

a. Equal Spacing. If the independent variable, time, is equally spaced (i.e., the difference between consecutive times is constant), then nice computational formulas can be used to estimate first and second derivatives. For example, if $x(t)$ is a position function of time, then $y'(t_k)$ is approximately:

$$-\frac{1}{12}\left[x(t_{k+2}) - x(t_{k-2})\right] + 2/3\left[x(t_{k+1}) - x(t_{k-1})\right] \qquad \text{(Eq 21)}$$

where the t_i are equally spaced. This expression is exact for all polynomials of degree 2 or less if the differences of consecutive independent

variables are one. The restriction of differences all one can be overcome by appropriate scaling in the independent variable. This type of formula can be derived using either Fourier series or differentiating a least-squares best fit polynomial, a special case of subparagraphs b, c, and d, below. The polynomial is the polynomial of degree n (where n + 1 is less than the number of data points being fit), which minimizes the sum of the squares of the difference in the observed value and polynomial value at the points being fitted.

b. Unequal Spacing. This was the situation encountered in the ATMT study. The formulas in this case are not as simple as those in subparagraph a, above. The idea again is to fit a polynomial of degree n:

$$x(t) = a_0 + a_1 t + \dots + a_{n-1} t^{n-1} + a_n t^n \qquad \text{(Eq 22)}$$

to m points, $m > n + 1$, which minimizes the sum of the squares of the differences between the observed value and x(t) at the m points. The n and m are determined from considering the application. For example, the degree of polynomial n may be heuristically set by matching the number of major acceleration changes during the interval of m points with the number of sign changes possible with the second derivative of an nth degree polynomial on the interval of m points. For a given set of data and n, the larger the m (the number of points) the "smoother" the polynomial values and derivatives will be. Thus, m must be large enough to smooth the values so that the velocities and accelerations are within the capability of the vehicle system. An upper bound on m possibly can be determined by checking if values are smoothed below known actual performance or by an autocorrelation program on the data, with trend and any discernible cycles removed to determine when the errors become essentially linearly independent. In the ATMT analysis, the degree used was 3 and the number of points was between 9 and 51. Different values would probably be needed for aircraft or other vehicles. The general approach with a large quantity of data is to fit a polynomial to each t_k, where $(m + 1)/2 \leq k \leq$ maximum number of points minus $(m - 1)/2$, and use the coefficients to compute the smoothed position, velocity, and acceleration in each component as a function of time. Thus, the problem in the final determination of m is to balance realistic velocity and acceleration with an analysis of the distances between the observed and smoothed positions. A complete statistical analysis of residuals would require an immense amount of computer time.

c. Derivation of Coefficients. The following notation is used in this discussion:

$x(t)$ = the actual x-coordinate (unknown)

$\bar{x}(t)$ = the observed x-coordinate

$\hat{x}(t)$ = the smoothed x-coordinate

$$\hat{x}(t) = a_0 + a_1 t + \ldots + a_n t^n = \sum_{j=0}^{n} a_j t^j \qquad \text{(Eq 23)}$$

m = number of points to be fitted

L = $(m - 1)/2$

t_k = the time in question: $(m + 1)/2 \leq k \leq$ maximum number of points minus $(m - 1)/2$

$$\hat{x}(t_k) = \sum_{j=0}^{n} a_j (t_k)^j$$

$$\hat{x} \text{ component of velocity at } t_k = \sum_{j=1}^{n} a_j j (t_k)^{j - 1}$$

$$\hat{x} \text{ component of acceleration at } t_k = \sum_{j=2}^{n} a_j j(j - 1)(t_k)^{j - 2}$$

The objective in least squares fit is to determine the coefficients a_p that minimize:

$$H(a_0, a_1, \ldots a_n) = \sum_{i=k-L}^{k+L} \left[\bar{x}(t_i) - \sum_{j=0}^{n} a_j (t_i)^j \right]^2. \qquad \text{(Eq 24)}$$

To find the coefficients a_p by minimizing H, the partial derivative of H with respect to each a_p must be calculated, the resulting expressions set equal to zero, and the a_p's solved for each p=0, 1, . . . n. In matrix form the solution is:

$$A = B^{-1}R$$

where:

$$A = (a_p)$$

$$R = (r_p)$$

$$B = (b_{pj}) \qquad p=0, 1, \ldots n,$$

$$r_p = \sum_{i=k-L}^{k+L} \bar{x}(t_i)(t_i)^p \qquad \text{(Eq 25)}$$

$$j=0, 1, \ldots n,$$

$$b_{pj} = \sum_{i=k-L}^{k+L} (t_i)^{p+j}.$$

d. Cautions and Comments. The equations of the previous paragraph must be applied with care when programmed on a computer. If the size of t_i is large relative to $t_i - t_{i-1}$, then it is advantageous to translate the t_i's so that $t_k = 0$ in order to improve numerical accuracy. After a translation it can be observed that if the t_i were equally spaced, then B (already a symmetric matrix) is a constant matrix, a fact that could be used to shorten the number of computations. If an orthogonal set of polynomials could be used (only reasonably possible in the case of equal spacing) the matrix B would be diagonal, and formulas of the type mentioned in subparagraph a above would be obtained. Table 4 contains the result of a trial with n=3, m=15. Figure 4 shows a plot of data joined by straight lines and a plot of smoothed points joined by straight lines in the x-y plane. Figure 5 is a plot of resultant velocity and resultant acceleration from the smoothed values. Further information is contained in Ralston.

Table 4. Least squares fit using 15 points, degree 3

t(sec)	$\hat{x}$ (meters)	$\hat{y}$ (meters)	$\hat{z}$ (meters)	distance traveled	distance between observed & smoothed
37009.50	55188.278	79249.262	383.526	0	.1903
37009.60	55188.695	79248.961	383.475	.5181	.1149
37009.78	55189.771	79248.366	383.363	1.752	.1420
37009.83	55190.086	79248.197	383.332	2.112	.0351
37009.91	55190.594	79247.934	383.284	2.685	.0381

t(sec)	$v_{\hat{x}}$ (m/sec)	$v_{\hat{y}}$ (m/sec)	$v_{\hat{z}}$ (m/sec)	velocity resultant
37009.50	5.901	- 3.071	- 0.588	6.679
37009.60	5.868	- 3.163	- 0.598	6.694
37009.78	6.021	- 3.341	- 0.604	6.912
37009.83	6.233	- 3.435	- 0.580	7.141
37009.91	6.443	-3.438	- 0.644	7.332

t(sec)	$a_{\hat{x}}$ (m/sec^2)	$a_{\hat{y}}$ (m/sec^2)	$a_{\hat{z}}$ (m/sec^2)	acceleration resultant
37009.50	- 0.754	- 0.864	0.016	1.147
37009.60	1.625	- 0.994	- 0.125	1.909
37009.78	1.644	- 0.738	- 0.048	1.803
37009.83	1.385	- 0.612	- 0.054	1.516
37009.91	1.083	- 0.696	- 0.032	1.288

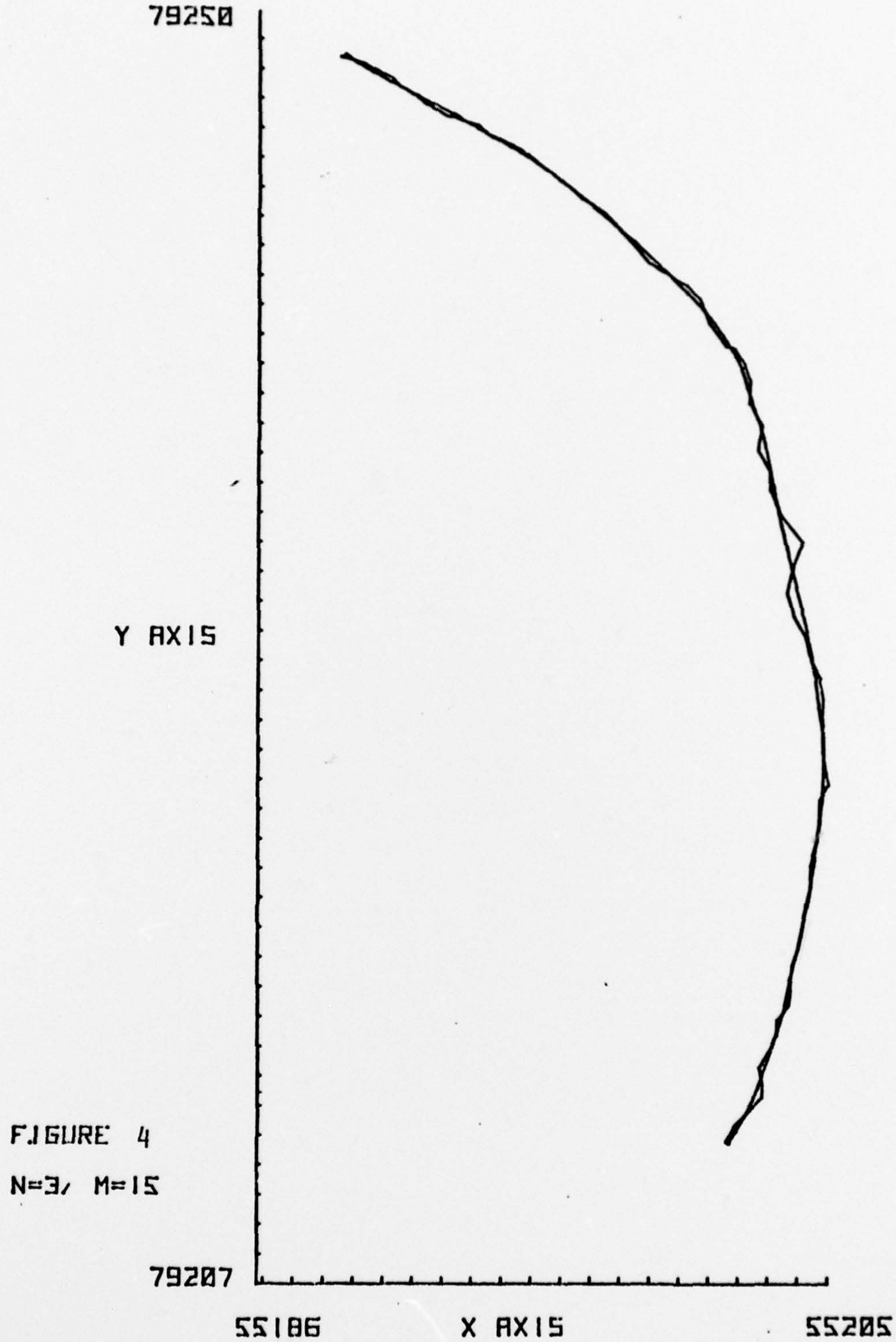

Figure 4. Data plot, n=3, m=15

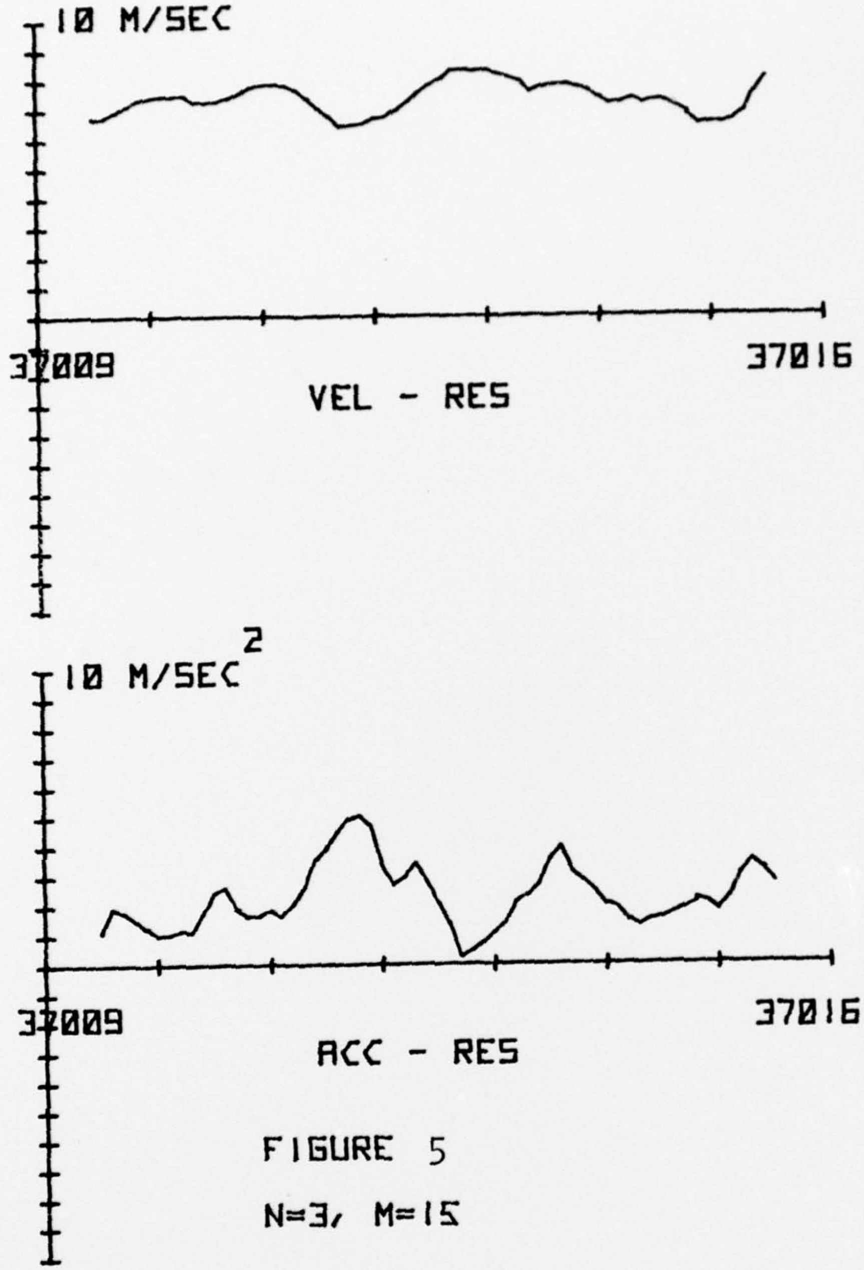

Figure 5. Smoothed velocity and acceleration plots

6. CONVOLUTION FILTERS.

a. Transforms. The objective in convolution filtering is to suppress the changes in the data caused by errors in the measuring system and not to change the desired information. In the ATMT study the position changed slowly in comparison to the errors. In this filtering, the original data are transformed by the z-transform from a function of time to a function of frequency. If T is the sampling interval (constant), N the number of samples, $\Omega = 2\pi/NT$, and $j = \sqrt{-1}$, then the transformed data are $X(k\Omega)$:

$$X(k\Omega) = \sum_{n=0}^{N-1} x(nT)e^{-j\Omega Tnk}, \quad k=0, 1, \ldots N-1.$$

Next, a function $H(k\Omega)$ is determined such that $H(k\Omega)$ is close to 1 for frequencies $2\pi k/NT$ near the frequencies of the position and $H(k\Omega)$ is close to 0 for frequencies $2\pi k/NT$ near the error frequencies. The selection of $H(k\Omega)$, the response function, is discussed in the next subparagraph. The functions $X(k\Omega)$ and $H(k\Omega)$ are multiplied so that the resulting function is approximately $X(k\Omega)$ for the frequencies of the position and approximately zero for the frequencies of the errors. To complete the filtering, the inverse z-transform of $H \cdot X$ is computed.

$$y(nT) = \frac{1}{N} \sum_{R=0}^{N-1} X(k\Omega)\, H(k\Omega)e^{j\Omega Tnk}$$

Thus, the main problem is to determine N and $H(k\Omega)$ to ensure $y(nT)$ is close to the true position part of $x(nT)$.

b. Response Functions. A large class of filters may be derived from $H(z)$ of the form:

$$H(z) = \frac{1}{\prod_{i=1}^{m} (1 - e^{-s_i T} z^{-1})}$$

These filters include Butterworth, Chebyshev, and Bessel filters for different choices of the s_i. The s_i for the Butterworth filter are located equally spaced on circle in the s complex plane of radius $w_c{}^J$, w_c being

the cutoff frequency of $|H(w_c{}^j)| \doteq .707$. The s_i for Chebyshev filter are located on an ellipse in the s-plane. Graphs of typical response functions H(z) are shown in figures 6 and 7.

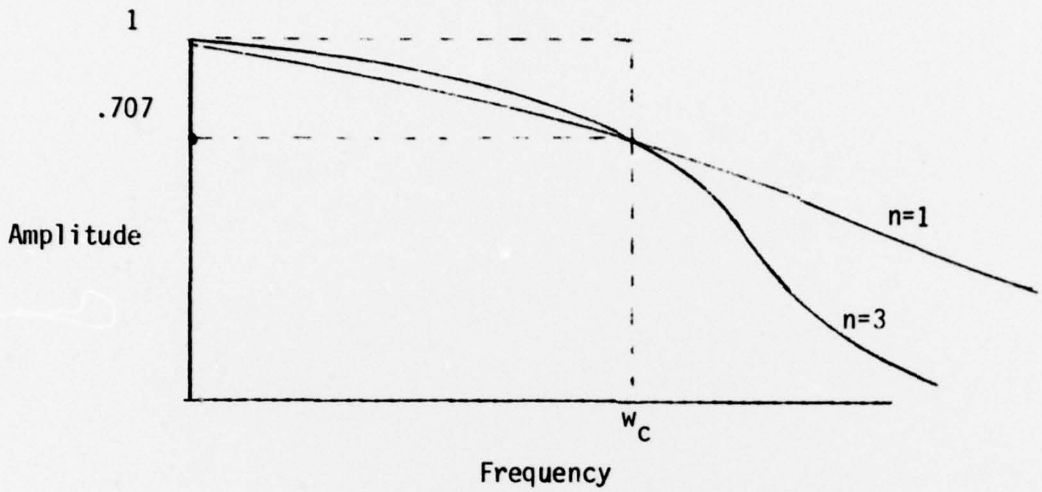

Figure 6. Butterworth Response Function $H(z)$ or $|F(jw)|^2 = \dfrac{1}{1 + (w/w_c)^{2n}}$

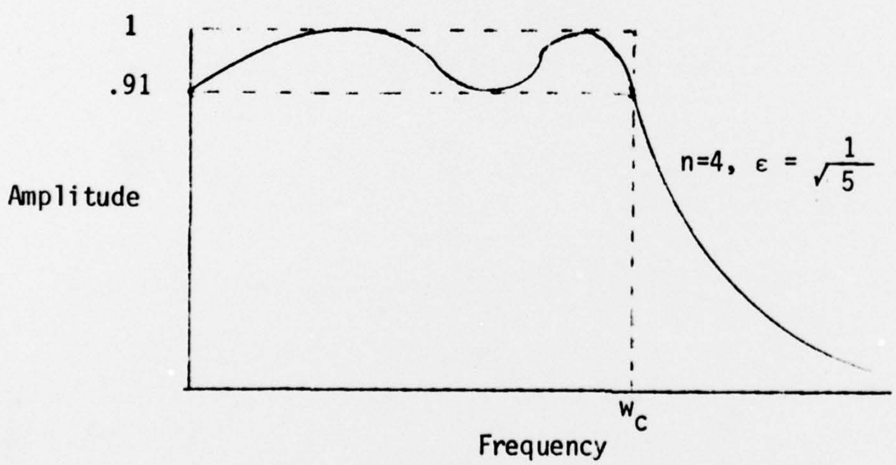

Figure 7. Chebyshev Response Function $H(z)$ or $|F(jw)|^2 \doteq \dfrac{1}{1 + \varepsilon^2 V_n{}^2 (w/w_c)}$,
V_n a Chebyshev polynomial of order n

Further development can be found in Gold and Rader, reference 5. Program versions of a third order Butterworth filter can be found in appendix B of AMSAA's TR151, reference 1.

7. MINIMUM MAXIMUM TECHNIQUES FOR FUNCTIONAL APPROXIMATION. This approach is generally applied when the value of a function needs to be approximated at a large number of domain points in some interval and it is not known a priori what these points will be. Any type of approximating functions (e.g., polynomials, rational functions, spline functions, exponential functions, other transcendental functions) can be used, but rational functions are the type used most often. The type is determined from a judgment of which type is most "like" the data being fitted as well as the computational ease and time considerations. Given the type of function to be used, the problem in this method is to choose the function of the given type that minimizes the maximum deviation between the observed values and the approximating function. For the ATMT application it was unreasonable to expect any one function type (except for the spline function approach) to be able to follow well a vehicle's path for 2 to 3 minutes without using large degree rational functions. Working the problem in subintervals or being concerned with evaluation at all points is similar to the problems discussed in paragraphs 4 and 5. For further reference, see Ralston, reference 8, or Rice, reference 9.

8. SUMMARY. The decision on which method is most appropriate for use in a given application must be based on judgment, weighing the considerations discussed in the previous paragraphs. Standard rules cannot be developed for these decisions; however, table 5 summarizes some of the characteristics of the various approaches that should be considered.

Table 5. Summary of method characteristics

	Better for Sparse (few) or Dense (many) Independent Variables	Computation Ease	Require Equal Spacing	Derivatives Readily Available at Points other than Data Points
Difference Quotients	Neither	Easy	No	No
Interpolation	Sparse	Hard	No	Yes
Kalman Filter	Dense	Hard	No	No
Spline	Sparse	Hard	No	Yes
Least Squares				
a. Equal Spaces	Either	Easy	Yes	No
b. Not Equal Spaces	Either	Hard	No	No
Convolution Filters	Either	Hard	No	No
Min Max Errors	Sparse	Hard	No	Depends on Function Used

9. REFERENCE LIST.

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This paper discusses several approaches to handling the filtering and smoothing of time related data. The specific case addressed is the Antitank Missile Test (ATMT) conducted at US Army Combat Developments Experimentation Command using the Range Measuring System. Methods discussed include difference quotients, Kalman filters, spline functions, least squares approximation, convolution filters, and minimum maximum error techniques.